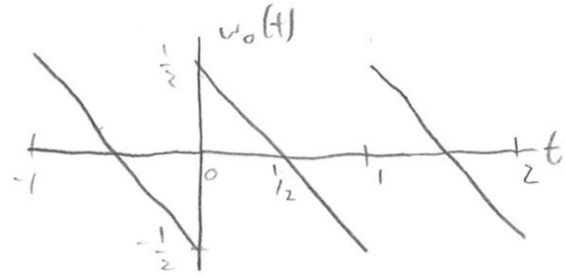
Bernoulli functions

Note: these are related to Bernoulli polynomials. cf. Wikipedia: Bernoulli polynomial

Let $w_0(t) = \frac{1}{2} - t$ on $(0,1)$ and $w_0(t+1) = w_0(t)$ (w_0 is 1-periodic).

It's easy to find $W_0[k]$ by direct integration:



$$W_0[k] = \int_0^1 \left(\frac{1}{2} - t\right) e^{-2\pi jkt}\,dt \quad (T_0 = 1)$$

$$= \frac{1}{2}\int_0^1 e^{-2\pi jkt}\,dt - \int_0^1 t e^{-2\pi jkt}\,dt$$

$$\int_0^1 e^{-2\pi jkt}\,dt = \left.\frac{e^{-2\pi jkt}}{-2\pi jk}\right]_{t=0}^{1} = -\frac{1}{2\pi jk}\left(e^{-2\pi jk} - e^{0}\right) = -\frac{1}{2\pi jk}(1-1) = 0$$

if $k \neq 0$

$$\int_0^1 t e^{-2\pi jkt}\,dt \qquad \begin{array}{ll} u = t & dv = e^{-2\pi jkt}\,dt \\ du = dt & v = \dfrac{e^{-2\pi jkt}}{-2\pi jk} \end{array}$$

$$= \left. t\,\frac{e^{-2\pi jkt}}{-2\pi jk}\right]_{t=0}^{1} + \int_0^1 \frac{e^{-2\pi jkt}}{2\pi jk}\,dt$$

$$= \left(\frac{e^{-2\pi jk}}{-2\pi jk} - 0\right) + \frac{1}{2\pi jk}\underbrace{\int_0^1 e^{-2\pi jkt}\,dt}_{=0} = \frac{-1}{2\pi jk} \quad \text{if } k \neq 0$$

$$W_0[0] = \int_0^1 w_0(t)\,dt = 0 \quad \text{(by inspection)}$$

$$W_0[k] = \begin{cases} \dfrac{1}{2\pi jk} & k \neq 0 \\ 0 & k = 0 \end{cases}$$

Now, we will define $w_n(t)$ so that $W_n[k]$ is very simple.

$$W_n[k] = \begin{cases} \dfrac{1}{(2\pi jk)^{n+1}} & k \neq 0 \\ 0 & k = 0 \end{cases}$$

$W_n[k] = 2\pi jk\, W_{n-1}[k]$ ← use derivative rule

$w_n(t) = \frac{d}{dt} w_{n-1}(t)$

$W_n[0] = 0$ ← use average value rule

$\int_0^1 w_n(t)\,dt = 0$

Bernoulli functions continued

We have constraints on $w_n(t)$, and we can generate all of them from $w_0(t)$.

$$w_1(t) = \int_0^t w_0(\lambda)\,d\lambda + c_1 \qquad (\text{so } w_1'(t) = w_0(t))$$

$$= \tfrac{1}{2}t - \tfrac{1}{2}t^2 + c_1$$

$$\int_0^1 w_1(t)\,dt = 0 = \int_0^1 \left(\tfrac{1}{2}t - \tfrac{1}{2}t^2 + c_1\right)dt = \tfrac{1}{4}t^2 - \tfrac{1}{6}t^3 + c_1 t\Big]_{t=0}^{1}$$

$$= \tfrac{1}{4} - \tfrac{1}{6} + c_1 = \tfrac{1}{12} + c_1 = 0 \rightarrow c_1 = -\tfrac{1}{12}$$

$$w_1(t) = -\tfrac{1}{2}t^2 + \tfrac{1}{2}t - \tfrac{1}{12}$$

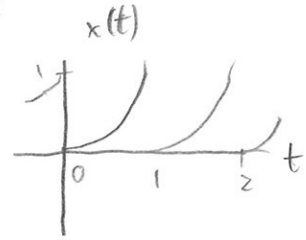
Here are some more:

$$w_2(t) = -\tfrac{1}{6}t^3 + \tfrac{1}{4}t^2 - \tfrac{1}{12}t$$

$$w_3(t) = -\tfrac{1}{24}t^4 + \tfrac{1}{12}t^3 - \tfrac{1}{24}t^2 + \tfrac{1}{720}$$

Notice that $w_n(t)$ is a linearly independent set of polynomials. This is obvious because w_n has degree $(n+1)$. So we can always write any polynomial as a linear combination of Bernoulli functions. We know the FS of each $w_n(t)$, and the Fourier transform is linear, so we now know the Fourier Series of all periodic extensions of polynomials!

Let $x(t) = t^3$, $t \in (0,1)$ and $x(t)$ is 1-periodic



$$x(t) = c_2 w_2(t) + c_1 w_1(t) + c_0 w_0(t) + x_0$$

equate coefficients and put in matrix form:

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{array}{c} \\ t^3 \\ t^2 \\ t \\ 1 \end{array} \begin{array}{c} \begin{array}{cccc} w_2 & w_1 & w_0 & x_0 \end{array} \\ \begin{bmatrix} -\frac{1}{6} & 0 & 0 & 0 \\ +\frac{1}{4} & -\frac{1}{2} & 0 & 0 \\ -\frac{1}{12} & +\frac{1}{2} & -1 & 0 \\ 0 & -\frac{1}{12} & +\frac{1}{2} & 1 \end{bmatrix} \end{array} \cdot \begin{bmatrix} c_2 \\ c_1 \\ c_0 \\ x_0 \end{bmatrix} \rightarrow \text{solve} \rightarrow \begin{array}{l} c_2 = -6 \\ c_1 = -3 \\ c_0 = \frac{3}{2} \\ x_0 = -1 \end{array}$$

Bernoulli functions cont.

$$x(t) = -6\,w_2(t) + -3\,w_1(t) + \tfrac{3}{2}w_0(t) + -1$$

$$X[k] = -6\,W_2[k] + -3\,W_1[k] + \tfrac{3}{2}W_0[k] - \delta[k]$$

$$= \begin{cases} \dfrac{-6}{(2\pi jk)^3} - \dfrac{3}{(2\pi jk)^2} + \dfrac{3}{2(2\pi jk)} & k \neq 0 \\ -1 & k = 0 \end{cases}$$

Find $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2}$.

Note: This is the "Basel problem", and when Euler was the first to solve it, it more or less put him on the map as a world class mathematician.

$$\sum_{k=-\infty}^{\infty} W_1[k] = \sum_{k\neq 0} \frac{1}{(2\pi jk)^2} = \frac{-1}{4\pi^2}\sum_{k\neq 0}\frac{1}{k^2} = \frac{-2}{4\pi^2}\sum_{k=1}^{\infty}\frac{1}{k^2} \quad \left(\frac{1}{(-k)^2} = \frac{1}{k^2}\right)$$

$$= w_1(0) = -\frac{1}{12}$$

$$\left(\frac{-1}{12}\right)\left(\frac{-4\pi^2}{2}\right) = \frac{\pi^2}{6} = \sum_{k=1}^{\infty}\frac{1}{k^2}$$

One more fact: $w_n(t)$ becomes more and more like scaled versions of $\sin(2\pi t)$ (n even) and $\cos(2\pi t)$ (n odd) as $n \to \infty$.

This makes sense if you consider:

$$W_n[k] = \begin{cases} \dfrac{1}{(2\pi jk)^{n+1}} & k \neq 0 \\ 0 & k = 0 \end{cases} \approx \begin{cases} \dfrac{1}{(2\pi jk)^{n+1}} & k = \pm 1 \\ 0 & k \neq \pm 1 \end{cases}$$

because all $W_n[k]$ other than $W_n[\pm 1]$ are much smaller. $W_n[\pm 2]$ is $\frac{1}{2^{n+1}}$ times smaller than $W_n[\pm 1]$, $W_n[\pm 3]$ is $\frac{1}{3^{n+1}}$ times smaller, etc. So:

$$w_n(t) \approx \begin{cases} \dfrac{(-1)^{n/2}}{(2\pi)^n}\,\dfrac{\sin(2\pi t)}{\pi} & n \text{ even} \\ \dfrac{(-1)^{\frac{n+1}{2}}}{(2\pi)^n}\,2\cos(2\pi t) & n \text{ odd} \end{cases}$$